HW 3

1.a

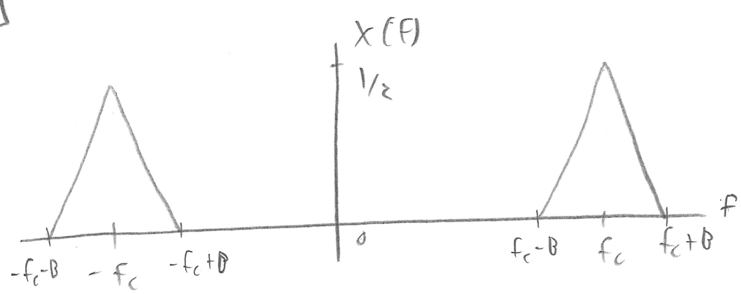
$$G(f) = \Lambda\left(\frac{f}{B}\right) \xrightarrow{\mathcal{F}^{-1}} g(t) = B\,\mathrm{sinc}^2(Bt)$$

$$\mathrm{Re}\left(g(t)e^{2\pi j f_c t}\right) = \frac{1}{2} g(t) e^{2\pi j f_c t} + \frac{1}{2} g^*(t) e^{-2\pi j f_c t}$$

$$= \frac{1}{2}\left(e^{2\pi j f_c t} + e^{-2\pi j f_c t}\right) g(t). \quad (g = g^*)$$

$$= \cos(2\pi f_c t)\, g(t). = \boxed{B\,\mathrm{sinc}^2(Bt)\cos(2\pi f_c t)}$$

1.b



$$\frac{1}{2}\left(X(f-f_c) + X(f+f_c)\right)$$

2

$$H(f) = \frac{1}{1+j\frac{f}{B}} = \frac{1 - jf/B}{1 + (f/B)^2} = \frac{1}{\sqrt{1+(f/B)^2}}\, e^{j\arctan(-f/B)}$$

$$|H| = \frac{1}{\sqrt{1+(f/B)^2}} \qquad \arg(H) = \arctan\left(\frac{f/B}{1}\right)$$

$Y(f) = H(f)X(f)$ and $|X(f)| \approx 0$ for $|f| > W$, $W \ll B$

So we're only interested in frequencies $|f| \ll B \rightarrow \frac{|f|}{B} \ll 1$

If $\left|\frac{f}{B}\right| \ll 1$, then $\arctan\left(-\frac{f}{B}\right) \approx -\frac{f}{B}$ and $\frac{1}{\sqrt{1+(f/B)^2}} \approx 1$

To see why, take the power series expansion:

$$\arctan(-x) = -x + \frac{x^3}{3} + \cdots \qquad \frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \cdots$$

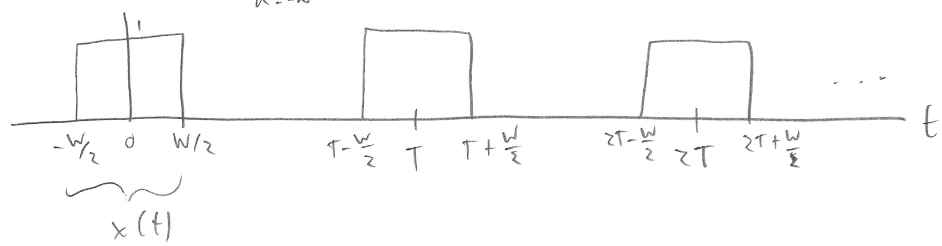
So $Y(f) \approx 1 \cdot e^{-j f/B} X(f)$. $\boxed{\text{This is just a delay:}}$

$$e^{-j f/B} X(f) = e^{-2\pi j f/2\pi B} X(f) \xrightarrow{\mathcal{F}^{-1}} x\left(t - \frac{1}{2\pi B}\right)$$

This is considered distortionless.

3] HW3

$$g(t) = \sum_{n=-\infty}^{\infty} g(t-nT)$$



We could integrate with the Fourier Series analysis formula:

$$G[k] = \frac{1}{T}\int_{-W/2}^{W/2} e^{-2\pi j k t/T}\,dt = \frac{1}{T}\left.\frac{e^{-2\pi j k t/T}}{-2\pi j k/T}\right]_{t=-\frac{W}{2}}^{\frac{W}{2}} \qquad (k \neq 0)$$

$$= -\frac{e^{-2\pi j k W/2T}}{2\pi j k} + \frac{e^{2\pi j k W/2T}}{2\pi j k}$$

$$= \frac{1}{2j}\left(e^{\pi j k W/T} - e^{\pi j k W/T}\right)\frac{1}{\pi k}$$

$$= \frac{\sin(\pi k W/T)}{\pi k} = \frac{W}{T}\operatorname{sinc}\left(k\frac{W}{T}\right)$$

By inspection, $G[0] = \frac{W}{T}$, so $\boxed{G[k] = \frac{W}{T}\operatorname{sinc}\left(k\frac{W}{T}\right)}$

We can also use the Poisson sum formula:

$$X(f) = W\operatorname{sinc}(Wf)$$

$$g(t) = \sum_{n=-\infty}^{\infty} x(t-nT) \rightarrow G[k] = \frac{1}{T}X\left(\frac{k}{T}\right) = \boxed{\frac{W}{T}\operatorname{sinc}\left(\frac{Wk}{T}\right)}$$

4.a]

If $x(t-\tau)$ is given to the system, we get $2x(t-\tau) - 3(x(t-\tau))^3$

If we delay the output, $y(t-\tau) = 2x(t-\tau) - 3(x(t-\tau))^3$ ← equal

4.b

$x(t) = A\cos(\omega_0 t)$

$y(t) = 2A\cos(\omega_0 t) - 3A^3\cos^3(\omega_0 t)$

$$\cos^3(\theta) = \left(\tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\right)^3 = \frac{1}{8}\left(e^{j3\theta} + 3e^{2j\theta}e^{-j\theta} + 3e^{j\theta}e^{-2j\theta} + e^{-3j\theta}\right)$$

(binomial expansion)

$$= \frac{1}{8}\left(e^{3j\theta} + e^{-3j\theta} + 3e^{j\theta} + 3e^{-j\theta}\right) = \frac{1}{4}\left(\cos(3\theta) + 3\cos(\theta)\right)$$

$$y(t) = 2A\cos(\omega_0 t) - \frac{9A^3}{4}\cos(\omega_0 t) - \frac{3A^3}{4}\cos(3\omega_0 t)$$

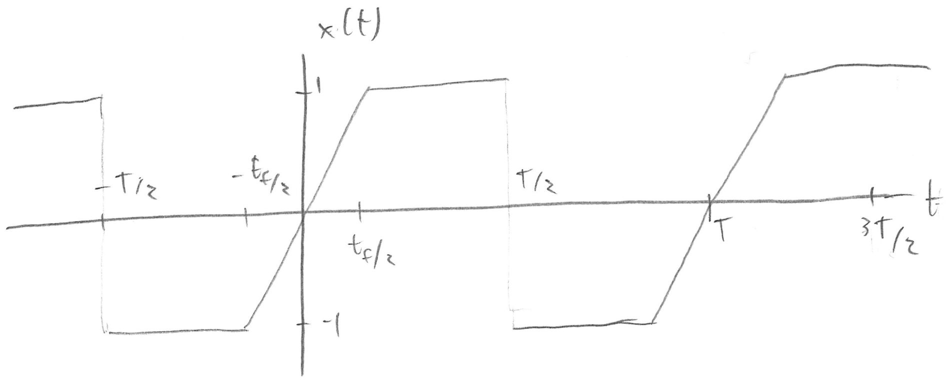
$$= \boxed{\left(2A - \frac{9A^3}{4}\right)\cos(\omega_0 t) - \frac{3A^3}{4}\cos(3\omega_0 t)}$$

$$\boxed{Y[k] = \begin{cases} A - \frac{9A^3}{8} & |k| = 1 \\ -\frac{3A^3}{8} & |k| = 3 \\ 0 & \text{else} \end{cases}}$$

either is OK

Notice how this nonlinear system produces new frequencies.

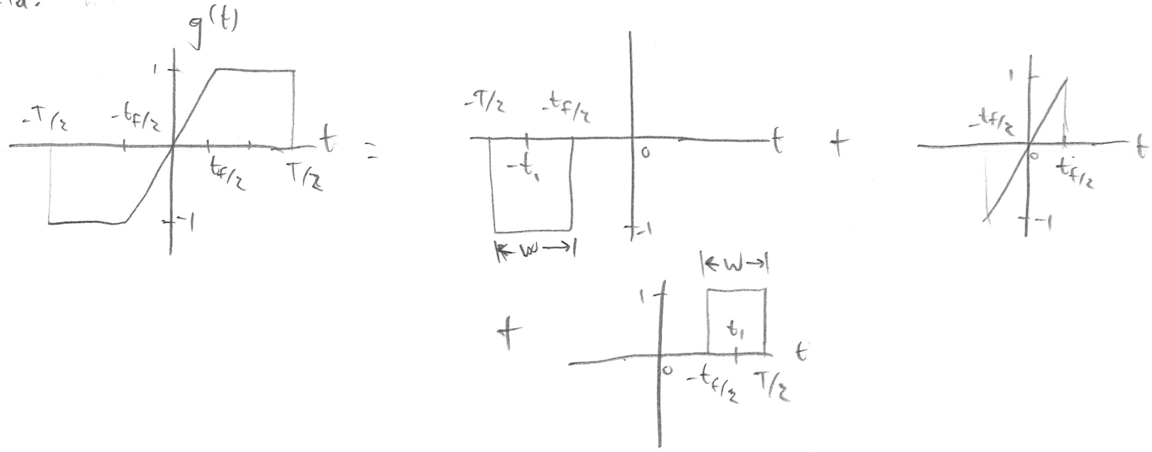
5.a



Note: I messed up this definition. It was supposed to be half-wave symmetric, and look more like this:  . Hopefully it still illustrates the point.

5.b)

Clearly there are many ways to do this. I'll do it with the Poisson sum formula:



$$g(t) = \frac{2t}{t_f}\Pi(t/t_f) + \Pi\left(\frac{t-t_1}{w}\right) + -\Pi\left(\frac{t+t_1}{w}\right)$$

$$w = \frac{T-t_f}{2}$$

$$t_1 = \frac{t_f}{2} + \frac{w}{2} = \frac{t_f}{2} + \frac{T}{4} - \frac{t_f}{4} = \frac{t_f+T}{4}$$

$$G(f) = \frac{2}{t_f} 2\pi j f \frac{d}{df}\left(t_f \operatorname{sinc}(f t_f)\right) + e^{-2\pi j f t_1} w \operatorname{sinc}(fw) - e^{2\pi j f t_1} w \operatorname{sinc}(fw)$$

$$\frac{d}{df}\operatorname{sinc}(f) = \pi\left(\frac{\cos(\pi f)}{\pi f} - \frac{\sin(\pi f)}{(\pi f)^2}\right) = \frac{1}{f}\left(\cos(\pi f) - \operatorname{sinc}(f)\right)$$

$$G(f) = \frac{2}{t_f} 2\pi j f\, t_f^2 \left(\frac{1}{f t_f}\left(\cos(\pi f t_f) - \operatorname{sinc}(f t_f)\right)\right) - \left(e^{2\pi j f t_1} - e^{-2\pi j f t_1}\right) w \operatorname{sinc}(wf)$$

$$= 4\pi j\left(\cos(\pi t_f f) - \operatorname{sinc}(f t_f)\right) - 2jw \sin(2\pi t_1 f)\operatorname{sinc}(wf)$$

$$x(t) = \sum_{n=-\infty}^{\infty} g(t-nT) \rightarrow X[k] = \frac{1}{T} G\left(\frac{k}{T}\right) =$$

$$\boxed{\frac{4\pi j}{T}\left(\cos(\pi k t_f/T) - \operatorname{sinc}(\pi k t_f/T)\right) - \frac{2jw}{T}\sin(2\pi t_1 k/T)\operatorname{sinc}(wk/T)}$$

5.c

HW3

From problem 3, a square wave of duty cycle $\frac{W}{T}$ has Fourier series coefficients of $\frac{W}{T} \operatorname{sinc}\left(\frac{Wk}{T}\right)$, which is $\frac{1}{2} \operatorname{sinc}\left(\frac{k}{2}\right)$ for 50% duty cycle.

This wave has a DC component and is phase shifted, but we're interested in the behavior of $|X[k]|$ as $k \to \infty$, so neither of these differences matter.